



Parameterized preconditioning for generalized saddle point problems arising from the Stokes equation[☆]

Zheng Li^{*}, Tie Zhang, Chang-Jun Li

Department of Mathematics, Northeastern University, Shenyang 110004, PR China

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ABSTRACT

A parameterized preconditioning framework is proposed to improve the conditions of the generalized saddle point problems. Based on the eigenvalue estimates for the generalized saddle point matrices, a strategy to minimize the upper bounds of the spectral condition numbers of the matrices is given, and the explicit expression of the quasi-optimal preconditioning parameter is obtained. In numerical experiment, parameterized preconditioning techniques are applied to the generalized saddle point problems derived from the mixed finite element discretization of the stationary Stokes equation. Numerical results demonstrate that the involved preconditioning procedures are efficient.

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1. Introduction

Generalized saddle point problems arise from many scientific and engineering applications, such as mixed finite element methods, constrained optimization, constrained least square problems, image processing, optimal control and so on (see [1]), and usually generate the linear systems in the following form:

$$\begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}, \quad (1.1)$$

the coefficient matrix

$$W = \begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \quad (1.2)$$

is called the generalized saddle point matrix, where $A \in \mathbb{R}^{m \times m}$ is symmetric and positive definite, $C \in \mathbb{R}^{n \times n}$ is symmetric and semi-positive definite, $B \in \mathbb{R}^{m \times n}$, $m \geq n$, and the Schur complement matrix $S = C + B^T A^{-1} B$ is positive definite (see [1,2]).

A large amount of research work has been devoted to the iterative methods for solving the large scale saddle point problems. Based on the splitting of the matrix W , researchers have developed various stationary iterative methods such

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^{*} Corresponding author.

E-mail addresses: neu_lizheng@hotmail.com, lizheng_mail@sina.com, lizheng-tiger@hotmail.com (Z. Li).

as Arrow–Hurwicz and Uzawa iterations (see [3]), the inexact Uzawa methods (see [3–6]), the generalized SOR methods (see [7–11]), the HSS method (see [12–14]) and so on. These methods have simple schemes and they are suitable to the parallel computation. Meanwhile, Krylov subspace methods usually have a high efficiency. The two-step CG method, the QMR method, the MINRES method and the GMRES method are introduced to solve the system (1.1) (see [1,3,15–19]). Specially, the MINRES method and the SYMMLQ method (see [19]) cater to the symmetric and indefinite systems and naturally become the candidate of the solvers of system (1.1).

In applications, all of these methods need efficient preconditioners to accelerate the convergence rates. However, establishing a practical preconditioner is usually difficult since the preconditioner is expected to have not only a significant efficiency but also a small computational cost and a clear mechanism. Besides, numerical computing experience teaches us that no method is omnipotent and each type of preconditioner has its own applicability. The main effort of this paper is building a simple and clear preconditioner that can improve the condition of the system (1.1) arising from some applications, such as the mixed finite element method for the stationary Stokes equation. Building such a preconditioner generally depends on the eigenvalue estimates for the generalized saddle point matrices. Fortunately, recently several references have studied the spectral properties of the generalized saddle point matrices (see [2,20–25]), which we believe very important and helpful to establishing the efficient preconditioners.

In this paper, we study the strategy of parameterized preconditioning for the generalized saddle point problems. We attempt to multiply the submatrices by some parameters to precondition the system (1.1). In the theoretical analysis, we give the upper bounds of the spectral condition numbers of the generalized saddle point matrices. Via a primary derivation we minimize the upper bounds of the spectral condition numbers, and then obtain the explicit expression of the quasi-optimal preconditioning parameters. The parameters can be adjusted to the optimum point so that the conditions of the systems may be improved significantly.

The remainder of this paper is organized as follows. In Section 2, we study the framework of parameterized preconditioning and obtain the quasi-optimal choice of the preconditioning parameters, and then give the corresponding preconditioning procedure. In Section 3, based on the different eigenvalue estimates we respectively give two types of preconditioning procedures for the special case $C = 0$. In Section 4, we apply the parameterized preconditioning techniques to the systems derived from the mixed finite element discretization of the stationary Stokes equation, and present the numerical results. Finally in Section 5, we give our conclusions.

2. Main results

It is well-known that the smaller condition number of the system may bring the more efficient solution. For instance, for the MINRES method we have the following result (see [19, p 56]):

$$\|r^{(m)}\| \leq \frac{1}{T_{[m/2]}(\theta_W)} \|r^{(0)}\|, \quad (2.1)$$

where $r^{(m)}$ is the m th iteration residual and $T_m(\lambda)$ is Chebyshev polynomial of m order, and

$$\theta_W = \frac{\kappa^2 + 1}{\kappa^2 - 1},$$

$[\cdot]$ is the integer function, and κ denotes the spectral condition number of the matrix W , which is the coefficient matrix of the linear system. Inequality (2.1) implies that the MINRES method converges faster if the spectral condition number κ becomes smaller.

In the following discussion, we usually use notation $\lambda_i(\cdot)$ to represent the i th eigenvalue of the corresponding matrix, and specially the largest eigenvalue is denoted by $\lambda_1(\cdot)$. We also use notation $\kappa(\cdot)$ to represent the spectral condition number of the corresponding matrix.

It is evident that the matrix W defined by (1.2) can be factorized as

$$W = \begin{pmatrix} I_m & 0 \\ B^T A^{-1} & I_n \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -S \end{pmatrix} \begin{pmatrix} I_m & A^{-1}B \\ 0 & I_n \end{pmatrix},$$

which implies that W is symmetric and indefinite. Hence all the eigenvalues of W are real numbers and located on both sides of the origin. According to the definition of the spectral condition number, it is easy to get the following result.

Lemma 2.1. Let H be a symmetric and indefinite matrix, and all the eigenvalues of H are located in the interval $I \equiv [l^-, r^-] \cup [l^+, r^+]$, where $l^- < r^- < 0 < l^+ < r^+$, then we have

$$\kappa(H) \leq \frac{\max\{r^+, -l^-\}}{\min\{l^+, -r^-\}} = \max\left\{\frac{r^+}{l^+}, \frac{r^+}{-r^-}, \frac{-l^-}{l^+}, \frac{l^-}{r^-}\right\}.$$

Let the submatrices of W be multiplied by some parameters, then system (1.1) can be preconditioned as follows:

$$\begin{pmatrix} \alpha A & \sqrt{\alpha} B \\ \sqrt{\alpha} B^T & -C \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} x \\ \frac{1}{\sqrt{\alpha}} y \end{pmatrix} = \begin{pmatrix} b \\ \frac{1}{\sqrt{\alpha}} q \end{pmatrix}, \quad (2.2)$$

where the preconditioning parameter $\alpha \in \mathbb{R}^+$. The coefficient matrix of (2.2) is denoted by

$$W(\alpha) = \begin{pmatrix} \alpha A & \sqrt{\alpha} B \\ \sqrt{\alpha} B^T & -C \end{pmatrix}. \quad (2.3)$$

Our purpose is to reduce $\kappa(W(\alpha))$ to a low value by adjusting the parameter α . The similar idea appeared in some references and sometimes called “scaling method” or “balance method” (see [1,2,13]). In this paper, we study the quasi-optimal choice of the preconditioning parameter α in detail.

Lemma 2.1 gives an upper bound of $\kappa(H)$. The information depends on the interval estimate for eigenvalues. Fortunately, some references gave the detailed eigenvalue estimates for matrix W ; one of the results is as follows.

Theorem 2.2 (Axelsson et al. [2]). Let matrix W be defined by (1.2), and its submatrix A and the Schur complement $S = C + B^T A^{-1} B$ be symmetric and positive definite. Let $0 < a_m \leq a_{m-1} \leq \dots \leq a_1$ and $0 \leq \sigma_n \leq \sigma_{n-1} \leq \dots \leq \sigma_1$ be the eigenvalues of A and $B^T A^{-1} B$ respectively, and let $\rho = \rho(S^{-1/2} B^T A^{-1} B S^{-1/2})$ be the spectral radius of the matrix $S^{-1/2} B^T A^{-1} B S^{-1/2}$. Let s_n and s_1 be the minimum and maximum eigenvalues of S respectively, and let $\Lambda(W) = \{\lambda_i(W) | i = 1, 2, \dots, m+n\}$ be the spectrum of W , then we have

$$\Lambda(W) \subseteq I \equiv [l^-, r^-] \cup [l^+, r^+],$$

where

$$l^- = -s_1, \quad r^- = \frac{-s_n}{1 + \frac{\rho s_n}{a_m}},$$

and

$$l^+ = a_m, \quad r^+ = a_1 + \sigma_1.$$

By Theorem 2.2, we get the following result.

Corollary 2.3. Let matrix $W(\alpha)$ be defined by (2.3), and $A, B, S, a_1, a_m, \sigma_1, \sigma_n, s_1, s_n, \rho$ be defined by Theorem 2.2, and $\Lambda(W(\alpha))$ be the spectrum of $W(\alpha)$, then we have

$$\Lambda(W(\alpha)) \subseteq I(\alpha) \equiv [l^-, r^-(\alpha)] \cup [l^+(\alpha), r^+(\alpha)],$$

where

$$l^- = -s_1, \quad r^-(\alpha) = \frac{-s_n}{1 + \frac{\rho s_n}{\alpha a_m}},$$

and

$$l^+(\alpha) = \alpha a_m, \quad r^+(\alpha) = \alpha a_1 + \sigma_1.$$

According to Lemma 2.1 and Corollary 2.3, we obtain the following conclusion.

Theorem 2.4. Let $A, B, S, a_1, a_m, \sigma_1, \sigma_n, s_1, s_n, \rho$ and $W(\alpha)$ be defined by Corollary 2.3, and let the spectral condition number of $W(\alpha)$ be denoted by $\kappa(W(\alpha))$, then

$$\kappa(W(\alpha)) \leq F(\alpha), \quad (2.4)$$

where

$$F(\alpha) = \max \{f_1(\alpha), f_2(\alpha), f_3(\alpha), f_4(\alpha)\}, \quad (2.5)$$

and

$$f_1(\alpha) = \frac{r^+(\alpha)}{l^+(\alpha)} = \frac{a_1}{a_m} + \frac{\sigma_1}{\alpha a_m}, \quad (2.6)$$

$$f_2(\alpha) = \frac{r^+(\alpha)}{-r^-(\alpha)} = \frac{\alpha a_1}{s_n} + \frac{\sigma_1}{s_n} + \frac{a_1 \rho}{a_m} + \frac{\rho \sigma_1}{\alpha a_m}, \quad (2.7)$$

$$f_3(\alpha) = \frac{-l^-}{l^+(\alpha)} = \frac{s_1}{\alpha a_m}, \quad (2.8)$$

$$f_4(\alpha) = \frac{l^-}{r^-(\alpha)} = \frac{\rho s_1}{\alpha a_m} + \frac{s_1}{s_n}. \quad (2.9)$$

$F(\alpha)$ defined in (2.4) is actually an upper bound of $\kappa(W(\alpha))$. To study the properties of $F(\alpha)$, we prepare the following lemmas.

Lemma 2.5 (Wang et al. [26, p 87]). Let $H \in \mathbb{C}^{n \times n}$ and $G \in \mathbb{C}^{n \times n}$ be both Hermitian matrices, then

$$\lambda_i(H) + \lambda_n(G) \leq \lambda_i(H + G), \quad i = 1, 2, \dots, n.$$

Lemma 2.6. Assume that H and G are both symmetric and semi-positive definite matrices, and matrix $H + G$ is positive definite, then for the spectral radius of the matrix $(H + G)^{-1}H$, denoted by $\rho((H + G)^{-1}H)$, we have

$$\rho((H + G)^{-1}H) \leq 1.$$

Lemma 2.5 and Lemma 2.6 lead to the following lemma.

Lemma 2.7. Assume that $A, B, S, \sigma_1, s_1, \rho$ are defined by Theorem 2.2, then we have

$$\sigma_1 \leq s_1, \quad (2.10)$$

and

$$\rho \leq 1. \quad (2.11)$$

Based on Theorem 2.4 and Lemma 2.7 and via a primary derivation, we directly get the following results.

Lemma 2.8. Let functions $f_1(\alpha), f_2(\alpha), f_3(\alpha)$ and $f_4(\alpha)$ be defined by (2.6)–(2.9), then we have the following conclusions.

- (i) $f_1(\alpha), f_3(\alpha), f_4(\alpha)$ are all decreasing functions of α .
- (ii) $f_2(\alpha)$ is increasing when $\alpha > \alpha_2^*$ and decreasing when $\alpha \leq \alpha_2^*$, here

$$\alpha_2^* = \sqrt{\frac{\rho \sigma_1 s_n}{a_1 a_m}}. \quad (2.12)$$

(iii) Equation

$$f_2(\alpha) = f_1(\alpha) \quad (2.13)$$

has two roots. The non-negative root is

$$\alpha_{2,1}^+ = \frac{s_n(1 - \rho)}{a_m}, \quad (2.14)$$

and the non-positive root is

$$\alpha_{2,1}^- = \frac{-\sigma_1}{a_1}.$$

Equation

$$f_2(\alpha) = f_3(\alpha) \quad (2.15)$$

has two roots. The non-negative root is

$$\alpha_{2,3}^+ = \frac{s_n}{2a_1} \left[-\left(\frac{\sigma_1}{s_n} + \frac{a_1 \rho}{a_m} \right) + \sqrt{\left(\frac{\sigma_1}{s_n} + \frac{a_1 \rho}{a_m} \right)^2 + \frac{4a_1(s_1 - \rho \sigma_1)}{s_n a_m}} \right], \quad (2.16)$$

and the non-positive root is

$$\alpha_{2,3}^- = \frac{s_n}{2a_1} \left[- \left(\frac{\sigma_1}{s_n} + \frac{a_1 \rho}{a_m} \right) - \sqrt{\left(\frac{\sigma_1}{s_n} + \frac{a_1 \rho}{a_m} \right)^2 + \frac{4a_1(s_1 - \rho\sigma_1)}{s_n a_m}} \right].$$

Equation

$$f_2(\alpha) = f_4(\alpha) \quad (2.17)$$

has two roots. The non-negative root is

$$\alpha_{2,4}^+ = \frac{s_1 - \sigma_1}{a_1}, \quad (2.18)$$

and the non-positive root is

$$\alpha_{2,4}^- = \frac{-\rho s_n}{a_m}.$$

Based on above discussion, now we give the following main result.

Theorem 2.9. Let $F(\alpha)$ be defined by (2.5), and let

$$\alpha^* = \max \{ \alpha_2^*, \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+ \}, \quad (2.19)$$

where $\alpha_2^*, \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+$ are defined by (2.12), (2.14), (2.16) and (2.18), then we have

$$F(\alpha^*) = \min_{\alpha} F(\alpha). \quad (2.20)$$

Proof. Observing the monotonicity of the functions $f_1(\alpha), f_3(\alpha), f_4(\alpha)$ and $f_2(\alpha)$, and note that all the equations, that is, (2.13), (2.15) and (2.17), have two roots, we get the following results.

$$\begin{cases} f_2(\alpha) \leq f_1(\alpha), & \alpha_{2,1}^- \leq \alpha \leq \alpha_{2,1}^+; \\ f_2(\alpha) > f_1(\alpha), & \text{otherwise.} \end{cases}$$

$$\begin{cases} f_2(\alpha) \leq f_3(\alpha), & \alpha_{2,3}^- \leq \alpha \leq \alpha_{2,3}^+; \\ f_2(\alpha) > f_3(\alpha), & \text{otherwise.} \end{cases}$$

$$\begin{cases} f_2(\alpha) \leq f_4(\alpha), & \alpha_{2,4}^- \leq \alpha \leq \alpha_{2,4}^+; \\ f_2(\alpha) > f_4(\alpha), & \text{otherwise.} \end{cases}$$

Therefore, for the envelop curve $F(\alpha)$ defined by (2.5) we have

$$F(\alpha) = \begin{cases} f_2(\alpha), & \alpha < \min \{ \alpha_{2,1}^-, \alpha_{2,3}^-, \alpha_{2,4}^- \}; \\ \max \{ f_1(\alpha), f_3(\alpha), f_4(\alpha) \}, & \min \{ \alpha_{2,1}^-, \alpha_{2,3}^-, \alpha_{2,4}^- \} \leq \alpha \leq \max \{ \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+ \}; \\ f_2(\alpha), & \alpha > \max \{ \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+ \}. \end{cases} \quad (2.21)$$

It is obvious that

$$\min \{ \alpha_{2,1}^-, \alpha_{2,3}^-, \alpha_{2,4}^- \} \leq 0 \leq \alpha_2^*,$$

thus $f_2(\alpha)$ is decreasing when $\alpha < \min \{ \alpha_{2,1}^-, \alpha_{2,3}^-, \alpha_{2,4}^- \}$. Therefore, it follows from (2.21) and Lemma 2.8 that $F(\alpha)$ is decreasing when

$$\alpha \leq \max \{ \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+ \}. \quad (2.22)$$

Next we observe the monotonicity of $F(\alpha)$ for the case

$$\alpha > \max \{ \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+ \}. \quad (2.23)$$

In case (2.23), it follows from the conclusion (ii) of Lemma 2.8 and (2.21) that $F(\alpha)$ reaches the minimum value at α_2^* if

$$\alpha_2^* > \max \{ \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+ \}.$$

Moreover, according to the conclusion (ii) of Lemma 2.8 we know $F(\alpha)$ is increasing for the case (2.23) if

$$\alpha_2^* < \max \{ \alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+ \}.$$

Therefore, combining the discussion on the cases (2.22) and (2.23) we conclude that $F(\alpha)$ reaches the minimum value at α^* defined by (2.19), completing this proof. \square

Theorem 2.9 gives an explicit expression of the quasi-optimal preconditioning parameter for matrix $W(\alpha)$. If we use (2.19) to determine the parameter $\alpha = \alpha^*$, then $F(\alpha)$, the upper bound of $\kappa(W(\alpha))$, can reach the minimum value, which usually means that $\kappa(W(\alpha))$ is sharply reduced and the condition of $W(\alpha)$ is improved. The procedure is briefly described as follows.

Procedure 2.1

Step 1: Input $a_1, a_m, s_1, s_n, \sigma_1, \rho$;

Step 2: Determine the parameter α^* by (2.19);

Step 3: Let $\alpha = \alpha^*$, and precondition the system (1.1) as system (2.2).

3. Results for the special case $C = O$

System (1.1) often has the following special form:

$$\begin{pmatrix} A & B \\ B^T & O \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix}. \quad (3.1)$$

The coefficient matrix of (3.1) is

$$\tilde{W} = \begin{pmatrix} A & B \\ B^T & O \end{pmatrix}, \quad (3.2)$$

where A is symmetric and positive definite, B has full column rank, and O denotes the zero matrix (see [1]). Similarly, the parameterized preconditioned system has the form

$$\begin{pmatrix} \alpha A & \sqrt{\alpha} B \\ \sqrt{\alpha} B^T & O \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ \frac{1}{\sqrt{\alpha}} q \end{pmatrix}, \quad (3.3)$$

and its coefficient matrix is

$$\tilde{W}(\alpha) = \begin{pmatrix} \alpha A & \sqrt{\alpha} B \\ \sqrt{\alpha} B^T & O \end{pmatrix}. \quad (3.4)$$

Let $A, B, S, s_1, s_n, \sigma_1, \sigma_n, \rho$ be defined by Theorem 2.2, then for the case $C = O$ it is evident that

$$S = B^T A^{-1} B,$$

$$s_1 = \sigma_1, \quad s_n = \sigma_n, \quad (3.5)$$

and

$$\rho(S^{-1/2} B^T A^{-1} B S^{-1/2}) = 1. \quad (3.6)$$

Let $\alpha_{2,1}^+, \alpha_{2,3}^+, \alpha_{2,4}^+$ and α_2^* be defined by (2.14), (2.16), (2.18) and (2.12), then it follows from (3.5) and (3.6) that

$$\alpha_2^* = \sqrt{\frac{s_1 s_n}{a_1 a_m}}, \quad (3.7)$$

$$\alpha_{2,1}^+ = \alpha_{2,3}^+ = \alpha_{2,4}^+ = 0. \quad (3.8)$$

For matrix $\tilde{W}(\alpha)$ defined by (3.4), based on (3.7), (3.8) and Theorem 2.9 we immediately get the following results.

Corollary 3.1. For the spectral condition number of $\tilde{W}(\alpha)$, we have

$$\kappa(\tilde{W}(\alpha)) \leq \tilde{F}(\alpha),$$

where

$$\begin{aligned} \tilde{F}(\alpha) &= \max \left\{ \frac{a_1}{a_m} + \frac{s_1}{\alpha a_m}, \frac{\alpha a_1 + s_1}{s_n} + \frac{a_1}{a_m} + \frac{s_1}{\alpha a_m}, \frac{s_1}{\alpha a_m}, \frac{s_1}{\alpha a_m} + \frac{s_1}{s_n} \right\} \\ &= \frac{\alpha a_1 + s_1}{s_n} + \frac{a_1}{a_m} + \frac{s_1}{\alpha a_m}, \end{aligned} \quad (3.9)$$

and $\tilde{F}(\alpha)$ reaches the minimum value at the point α^{**} , where

$$\alpha^{**} = \sqrt{\frac{s_1 s_n}{a_1 a_m}}. \quad (3.10)$$

A proper preconditioning parameter can minimize the upper bound of the condition number. However, we note that the different eigenvalue estimates may lead to the different upper bounds of the condition number. In fact, Ref. [20] gave another type of eigenvalue estimate for the matrix (3.2) as follows.

Theorem 3.2 (Rusten et al. [20], Benzi et al. [1]). Assume that matrix \tilde{W} is defined by (3.2), and its submatrices $A \in \mathbb{R}^{m \times m}$ are symmetric and positive definite, $B \in \mathbb{R}^{m \times n}$ has full column rank, $m \geq n$. Let a_m and a_1 be the minimum and maximum eigenvalues of A respectively, b_n and b_1 be the minimum and maximum singular values of B respectively, and the spectrum of \tilde{W} be denoted by $\Lambda(\tilde{W})$, then

$$\Lambda(\tilde{W}) \subseteq I \equiv [l^-, r^-] \cup [l^+, r^+],$$

where

$$l^- = \frac{1}{2} \left(a_m - \sqrt{a_m^2 + 4b_1^2} \right), \quad r^- = \frac{1}{2} \left(a_1 - \sqrt{a_1^2 + 4b_n^2} \right),$$

and

$$l^+ = a_m, \quad r^+ = \frac{1}{2} \left(a_1 + \sqrt{a_1^2 + 4b_1^2} \right).$$

Applying the parameterized preconditioning technique to system (3.1), the preconditioned system can be simplified as

$$\begin{pmatrix} \alpha A & B \\ B^T & O \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ \frac{1}{\alpha} q \end{pmatrix}, \quad (3.11)$$

and its coefficient matrix is denoted by

$$\overline{W}(\alpha) = \begin{pmatrix} \alpha A & B \\ B^T & O \end{pmatrix}. \quad (3.12)$$

Based on Theorem 3.2 and via a brief analysis, we can obtain the following result.

Theorem 3.3. Assume that $\overline{W}(\alpha)$ is defined by (3.12), and its spectral condition number is denoted by $\kappa(\overline{W}(\alpha))$. Let

$$G(\alpha) = \max \{g_1(\alpha), g_2(\alpha)\}, \quad (3.13)$$

where

$$g_1(\alpha) = \frac{\sqrt{\alpha^2 a_1^2 + 4b_1^2} + \alpha a_1}{\sqrt{\alpha^2 a_1^2 + 4b_n^2} - \alpha a_1}, \quad g_2(\alpha) = \frac{\sqrt{\alpha^2 a_1^2 + 4b_1^2} + \alpha a_1}{2\alpha a_m},$$

then we have

$$\kappa(\overline{W}(\alpha)) \leq G(\alpha).$$

Moreover, let

$$\alpha^{***} = \frac{b_n}{\sqrt{a_m^2 + a_1 a_m}}, \quad (3.14)$$

then we have

$$G(\alpha^{***}) = \min_{\alpha} G(\alpha).$$

Remark 3.1. The proof of Theorem 3.3 is elementary and here we omit it.

Therefore, based on the theoretical results of [Corollary 3.1](#) and [Theorem 3.3](#), we get two procedures to precondition the system (3.1). They are described as follows.

Procedure 3.1

Step 1: Input a_1, a_m, s_1, s_n ;

Step 2: Determine the parameter α^{**} by (3.10);

Step 3: Let $\alpha = \alpha^{**}$, and precondition the system (3.1) as (3.4)

Procedure 3.2

Step 1: Input a_1, a_m, b_n ;

Step 2: Determine the parameter α^{***} by (3.14);

Step 3: Let $\alpha = \alpha^{***}$, and precondition the system (3.1) as (3.12).

Remark 3.2. In applications it is usually not easy to obtain the precise eigenvalue information of the complicated large scale matrices like $S = C + B^T A^{-1} B$. We believe that the research on the fast eigenvalue algorithms for these matrices is necessary and important.

4. Numerical experiment

We observe the performance of the parameterized preconditioning procedures by the numerical experiments of the Stokes equation. We also solve the systems before and after preconditioning by the well-known MINRES method and SYMMLQ method, and the stop criterion is

$$\frac{\|r^{(k)}\|_2}{\|r^{(0)}\|_2} \leq 10^{-10},$$

where $r^{(k)}$ is the k th iterative residual. For instance, for the system (1.1)

$$r^{(k)} = \begin{pmatrix} b \\ q \end{pmatrix} - \begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} \begin{pmatrix} u^{(k)} \\ p^{(k)} \end{pmatrix}.$$

The initial guesses $u^{(0)}$ and $p^{(0)}$ are both zero vectors. The experiments are performed on Intel Core P8600 (CPU 2.4 GHz, RAM 2 GB), Windows XP system and MATLAB 7.0. The information of the eigenvalues and conditions of the matrices are all acquired by the MATLAB functions `eig(·)` and `cond(·)`.

The following stationary Stokes equation is a classical problem in computational fluid dynamics (see [1,2,6,27,28]).

$$\begin{aligned} -\Delta u + \nabla p &= f, \quad \text{in } \Omega, \\ -\operatorname{div} u &= 0, \quad \text{in } \Omega, \\ \int_{\Omega} p d\Omega &= 0, \\ u|_{\partial\Omega} &= 0, \end{aligned} \tag{4.1}$$

where $\Omega = (0, 1) \times (0, 1)$ is a unit square domain, and $\partial\Omega$ is the boundary of Ω . Vector u denotes the velocity, and p denotes the pressure. Mixed finite element methods are applied to discretizing Eq. (4.1) (see [6,27]). Different subdivision geometries will lead to the different linear systems. We test two types of mixed finite element, that is, the stabilized Q1-P0 element (see [6,27]) and the P1-P0 element (see [28]), to generate the linear systems.

4.1. Example I

In the first example, we use the stabilized Q1-P0 mixed finite element method to discretize Eq. (4.1) (see [6,27]). For the velocity, the domain Ω is partitioned into $ne \times ne$ uniform square elements. Let $h = 1/ne$. The resulting linear systems have the form as (1.1), and the coefficient matrices have the form

$$W = \begin{pmatrix} A & B \\ B^T & -C \end{pmatrix} = \begin{pmatrix} \hat{A} & B_1 \\ B_1^T & \hat{A} & B_2 \\ B_2^T & -C \end{pmatrix}.$$

The element matrices are described as follows.

$$\hat{A}_e = \frac{1}{6} \begin{pmatrix} 4 & -1 & -1 & -2 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ -2 & -1 & -1 & 4 \end{pmatrix}, \quad (B_1)_e = \frac{h}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad (B_2)_e = \frac{h}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Table 4.1

Numerical results of Example I by Procedure 2.1.

h	1/4	1/8	1/16	1/32
$\kappa(W)$	512.0859	1.1358e+004	2.0318e+005	3.3687e+006
Upp(1)	513.7335	1.1367e+004	2.0322e+005	3.3689e+006
MINRES, IT(1)	18	115	384	974
SYMMLQ, IT(1)	18	116	370	1415
α^*	0.1146	0.0279	0.0069	0.0017
$\kappa(W(\alpha^*))$	65.7007	360.9151	1.5968e+003	6.5920e+003
Upp(α^*)	67.7299	372.1450	1.6501e+003	6.8205e+003
MINRES, IT(α^*)	17	76	252	700
SYMMLQ, IT(α^*)	17	76	252	705

Table 4.2

Numerical results of Example II by Procedure 3.1.

h	1/4	1/8	1/16	1/28
$\kappa(W)$	215.5935	1.0710e+003	4.7128e+003	1.5031e+004
Upp(1)	228.5147	1.1330e+003	4.9838e+003	1.5895e+004
MINRES, IT(1)	19	122	304	566
SYMMLQ, IT(1)	19	122	317	567
α^{**}	0.0377	0.0312	0.0291	0.0284
$\kappa(W(\alpha^{**}))$	13.0436	44.0692	168.3617	510.4222
Upp(α^{**})	32.6861	135.0539	557.3183	1.7381e+003
MINRES, IT(α^{**})	19	93	225	409
SYMMLQ, IT(α^{**})	19	93	225	417

Matrix C is resulted from the global stabilization via the global jump formulation, and

$$C(u_h, p_h) = h \sum_{i=1}^{N_s} \int_{\partial\Omega_i} [u_h] [p_h] ds,$$

where $[\cdot]$ is the jump operator, and the summation is over all interior inter-element edges $\{\partial\Omega_i | i = 1, 2, \dots, N_s\}$. We apply Procedure 2.1 to preconditioning the systems. Table 4.1 shows the numerical results. The quasi-optimal preconditioning parameter α^* is determined by (2.19). The condition numbers of the original saddle point matrices (denoted by $\kappa(W)$) and the preconditioned saddle point matrices (denoted by $\kappa(W(\alpha^*))$) are compared. We also observe the upper bounds of the condition numbers before and after preconditioning, and they are represented by Upp(1) and Upp(α^*) respectively, where $\text{Upp}(1) = F(1)$, $\text{Upp}(\alpha^*) = F(\alpha^*)$, and $F(\alpha)$ is defined by (2.5). We use the MINRES method and the SYMMLQ method to solve the systems respectively. Notation $IT(1)$ means the iteration numbers for the original systems, and $IT(\alpha^*)$ means the iteration numbers for the preconditioned systems.

We can see from Table 4.1 that $\kappa(W(\alpha^*))$ are much smaller than $\kappa(W)$ especially when $h \rightarrow 0$, and consequently the convergence rate of the iterative methods increases after the preconditioning. The numerical results coincide with the theoretical analysis of Theorem 2.9.

4.2. Example II

In the second example, we use the P1-P0 mixed finite element method to discretize the Stokes equation (4.1). We divide Ω into uniform grids of triangular elements. Joining the midpoints of the edges on each triangle we partition each coarse triangle into four refined triangles, and let h denote the length of the right angle edge of each refined triangle. The details of the discretization and its theoretical consideration can be found in Ref. [28]. The coefficient matrices of the resulting linear systems have the following form:

$$\tilde{W} = \begin{pmatrix} A & B \\ B^T & O \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{A} & B_1 \\ B_1^T & \tilde{A} & B_2 \\ B_2^T & B_2^T & O \end{pmatrix},$$

where A is symmetric and positive definite, B has a full column rank.

Procedure 3.1 and Procedure 3.2 are applied to the systems respectively. The numerical results are reported in Table 4.2 and Table 4.3. The meaning of the notations is similar to Example I except that parameter α^{**} and α^{***} are determined by (3.10) and (3.14) respectively. Numerical results demonstrate that Procedure 3.1 and Procedure 3.2 are both appropriate for the Stokes equation. Similar to Example I, the procedures are efficient when $h \rightarrow 0$. In the case $h = 1/4$ of Table 4.3, it is abnormal that the iterative time (IT) increases after preconditioning. We attribute it to the effect of the rounding errors.

Table 4.3
Numerical results of Example II by Procedure 3.2.

h	1/4	1/8	1/16	1/28
$\kappa(W)$	215.5935	1.0710e+003	4.7128e+003	1.5031e+004
Upp(1)	2.5914e+003	3.7789e+004	5.9131e+005	5.4560e+006
MINRES, IT(1)	19	122	304	566
SYMMLQ, IT(1)	19	122	317	567
α^{***}	0.0888	0.0512	0.0265	0.0153
$\kappa(W(\alpha^{***}))$	7.1228	28.8105	198.5111	1.0131e+003
Upp(α^{***})	17.5021	99.2486	639.2077	3.0551e+003
MINRES, IT(α^{***})	20	77	204	417
SYMMLQ, IT(α^{***})	20	77	207	424

5. Conclusion

In this paper, we proposed a quasi-optimal parameterized preconditioning framework for the generalized saddle point problems. Determining the quasi-optimal parameters depends on the eigenvalue estimates given by Ref. [2,20], and the corresponding parameterized preconditioning procedures are developed. The numerical results demonstrate that the involved parameterized preconditioning procedures are usually efficient in applications, especially for the systems derived from the finite element methods for the Stokes equation. We have to note that these preconditioning procedures are feasible only when the estimate upper bounds Upp(1) are not far away from the condition numbers $\kappa(W)$. We also guess that the procedures may be improved provided that a more precise eigenvalue estimate is given. It is encouraging that researchers have present some new eigenvalue estimate for the saddle point matrices (see [24,25]). These valuable results may lead to a more satisfying procedure. The further research is underway.

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